A Sharpened Condition for Strict Log-Convexity of the Spectral Radius via the Bipartite Graph

Lee Altenberg
Associate Editor, *BioSystems*
altenber@hawaii.edu

Abstract

Friedland (1981) showed that for a nonnegative square matrix $\mathbf{A}$, the spectral radius $r(e^{\mathbf{D}}\mathbf{A})$ is a log-convex functional over the real diagonal matrices $\mathbf{D}$. He showed that for fully indecomposable $\mathbf{A}$, $\log r(e^{\mathbf{D}}\mathbf{A})$ is strictly convex over $\mathbf{D}_1, \mathbf{D}_2$ if and only if $\mathbf{D}_1 - \mathbf{D}_2 \neq c\,\mathbf{I}$ for any $c \in \mathbb{R}$. Here the condition of full indecomposability is shown to be replaceable by the weaker condition that $\mathbf{A}$ and $\mathbf{A}^\top\mathbf{A}$ be irreducible, which is the sharpest possible replacement condition. Irreducibility of both $\mathbf{A}$ and $\mathbf{A}^\top\mathbf{A}$ is shown to be equivalent to irreducibility of $\mathbf{A}^2$ and $\mathbf{A}^\top\mathbf{A}$, which is the condition for a number of strict inequalities on the spectral radius found in Cohen, Friedland, Kato, and Kelly (1982). Such 'two-fold irreducibility' is equivalent to joint irreducibility of $\mathbf{A}, \mathbf{A}^2, \mathbf{A}^\top\mathbf{A}$, and $\mathbf{A}\mathbf{A}^\top$, or in combinatorial terms, equivalent to the directed graph of $\mathbf{A}$ being strongly connected and the simple bipartite graph of $\mathbf{A}$ being connected. Additional ancillary results are presented.

Keywords: two-fold irreducible, strict convexity, superconvexity, sign pattern, chainable matrix, primitive matrix

MSC Subject Classification:

05C50 Graphs and linear algebra (matrices, eigenvalues, etc.),
15A42 Inequalities involving eigenvalues and eigenvectors,
15B35 Sign pattern matrices,
15B48 Positive matrices and their generalizations,
15A16 Matrix exponential and similar functions of matrices,
15A18 Eigenvalues, singular values, and eigenvectors.

1. Introduction

We begin with a theorem of Friedland on the log-convexity of the spectral radius of a nonnegative matrix ('superconvexity' as Kingman [18] called it).

Theorem 1 (Friedland Theorem 4.2 [10])**.** *Let $\mathcal{D}_n$ be the set of $n \times n$ real-valued diagonal matrices. Let $r(\mathbf{A})$ refer to the spectral radius of a matrix $\mathbf{A}$. Let $\mathbf{A}$*

Published in Linear Algebra and Its Applications*, DOI: 10.1016/j.laa.2013.01.008*

be a fixed $n \times n$ non-negative matrix having a positive spectral radius. Define $R\colon \mathcal{D}_n \to \mathbb{R}$ by $R(\mathbf{D}) := \log r(e^{\mathbf{D}}\mathbf{A})$. Then $R(\mathbf{D})$ is a convex functional on $\mathcal{D}_n$. Specifically: for each $\mathbf{D}_1, \mathbf{D}_2 \in \mathcal{D}_n$,

$$R((\mathbf{D}_1 + \mathbf{D}_2)/2) \leq (R(\mathbf{D}_1) + R(\mathbf{D}_2))/2. \tag{1}$$

Moreover, if $\mathbf{A}$ is irreducible and the diagonal entries of $\mathbf{A}$ are positive (or $\mathbf{A}$ is fully indecomposable) then equality holds in (1) if and only if

$$\mathbf{D}_1 - \mathbf{D}_2 = c\,\mathbf{I} \tag{2}$$

for some $c \in \mathbb{R}$, where $\mathbf{I}$ is the identity matrix.

In a recent paper, Cohen [7] asks whether a weaker condition may be substituted in the theorem for the condition that $\mathbf{A}$ be fully indecomposable. In particular, Cohen asks whether $\mathbf{A}$ being primitive would suffice.

Here, these questions are answered: yes — the condition that $\mathbf{A}$ is fully indecomposable can be weakened; but no — the condition that $\mathbf{A}$ be primitive is too weak. A condition is found in between these two that can be substituted in the theorem — that $\mathbf{A}^\top\mathbf{A}$ and $\mathbf{A}$ be irreducible — and it will be shown that this condition is the sharpest possible. The combination of irreducible $\mathbf{A}$ and $\mathbf{A}^\top\mathbf{A}$ is shown to be equivalent to the condition found for several strict inequalities in [8], which is that $\mathbf{A}^2$ and $\mathbf{A}^\top\mathbf{A}$ be irreducible.

Several ancillary results are also presented. Specific counterexamples are constructed for full indecomposability and primitivity: 1) partly decomposable matrices that nevertheless require $\mathbf{D}_1 - \mathbf{D}_2 = c\,\mathbf{I}$ for equality in (1), and 2) primitive matrices that produce equality in (1) even though $\mathbf{D}_1 - \mathbf{D}_2 \neq c\,\mathbf{I}$.

2. Main Question

In Theorem 1, the equality in (1) resulting from $\mathbf{D}_1 - \mathbf{D}_2 = c\,\mathbf{I}$ is readily verified for the 'if' direction. What is of interest is therefore the 'only if' direction. This is formalized as follows:

Definition 2 (Property I)**.** *A nonnegative $n \times n$ matrix $\mathbf{A}$ is said to* have 'Property I' *when*

1. *for $\mathbf{C}, \mathbf{D} \in \mathcal{D}_n$ and some $t \in (0, 1)$, the equality*

$$\log r(e^{(1-t)\mathbf{C}+t\mathbf{D}}\mathbf{A}) = (1 - t) \log r(e^{\mathbf{C}}\mathbf{A}) + t \log r(e^{\mathbf{D}}\mathbf{A}), \tag{3}$$

implies $\mathbf{C} - \mathbf{D}$ is a scalar matrix*, i.e.*

$$\mathbf{C} - \mathbf{D} = c\,\mathbf{I} \tag{4}$$

for some $c \in \mathbb{R}$;

2. *or equivalently, $\mathbf{C} - \mathbf{D}$ being nonscalar implies that, for all $t \in (0, 1)$,*

$$\log r(e^{(1-t)\mathbf{C}+t\mathbf{D}}\mathbf{A}) < (1 - t) \log r(e^{\mathbf{C}}\mathbf{A}) + t \log r(e^{\mathbf{D}}\mathbf{A}). \tag{5}$$

Irreducibility is central to Property I. Irreducibility, while being a very well-known property, has been defined in a number of equivalent ways, so these definitions are now given explicitly. First some notation needs to be described:

An $n \times n$ matrix is represented as $[A_{ij}]_{i,j=1}^n \equiv [A_{ij}]_{i,j \in \{1,\ldots,n\}} \equiv \mathbf{A}$.

The (i,j) element of matrix $\mathbf{A}$ is represented by $[\mathbf{A}]_{ij} \equiv A_{ij}$.

$\mathbf{A} > \mathbf{0}$, $\mathbf{x} > \mathbf{0}$ means all elements of matrix $\mathbf{A}$ or vector $\mathbf{x}$ are strictly positive.

$\mathbf{D} \equiv$ diag$[D_i]$ is a diagonal matrix with diagonal elements D_i.

Two equivalent properties are typically used to define irreducibility:

Definition 3 (Irreducibility, Definition 1 [11, p. 50])**.** *An $n \times n$ square matrix $\mathbf{A}$ is called* irreducible *if the index set $\{1, 2, \ldots, n\}$ cannot be partitioned into two nonempty sets $\mathcal{S}_1, \mathcal{S}_2$ such that*

$$A_{ij} = 0 \text{ for all } i \in \mathcal{S}_1, j \in \mathcal{S}_2.$$

Definition 4 (Irreducibility, Definition 2 [11, p. 50])**.** *An $n \times n$ square matrix $\mathbf{A}$ is called* irreducible *if there is no permutation matrix $\mathbf{P}$ such that*

$$\mathbf{A} = \mathbf{P} \begin{bmatrix} \mathbf{A}_1 & \mathbf{0}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_2 \end{bmatrix} \mathbf{P}^\top,$$

where $\mathbf{P}^\top$ is the transpose of $\mathbf{P}$, $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_{21}$ are $p \times p$, $q \times q$, and $q \times p$ matrices respectively, $\mathbf{0}_{12}$ is a $p \times q$ matrix of zeros, and $p + q = n$, $p, q \geq 1$.

Definition 5 (Reducibility)**.** *A square matrix is called* reducible *if it is not irreducible.*

For nonnegative matrices, Seneta [23, Definition 1.6, p. 18] (also used in [21, p. 61]) *defines* irreducibility in the following way, but as Gantmacher [11] shows, this is a consequence of the definitions above:

Corollary 6 (Irreducibility, Nonnegative Matrices [11, Corollary, p. 52])**.** *An $n \times n$ nonnegative matrix $\mathbf{A}$ is irreducible if for each pair (i,j) of its index set, there exists a positive integer $m \equiv m(i,j)$ such that $[\mathbf{A}^m]_{ij} > 0$.*

Equivalent to irreducibility is the following key property (usually stated as strong connectivity of the associated directed graph of a matrix, but stated more directly here).

Theorem 7 ([5, Theorem 3.2.1])**.** *A square matrix $\mathbf{A}$ is irreducible if and only if, for each pair of indices (i,j) there is a sequence of nonzero elements from i to j, $(A_{ih_1}, A_{h_1 h_2}, \ldots, A_{h_p,j})$ or $A_{ij} \neq 0$.*

Some additional properties are defined.

Definition 8 (Primitive)**.** *A nonnegative matrix* $\mathbf{A}$ *is called* primitive *if there is some positive integer* t *such that* $\mathbf{A}^t > \mathbf{0}$ *[23, Definition 1.1]. An irreducible matrix is called* imprimitive *if it is not primitive.*

Definition 9 (Index of Imprimitivity or Period)**.** *The* index of imprimitivity *or* period, $\gamma(\mathbf{A})$, *of a nonnegative irreducible matrix* $\mathbf{A}$ *is the greatest common divisor of the set* $\{t\colon [\mathbf{A}^t]_{ii} > 0, i = 1, \ldots, n\}$*. [23, Definition 1.3].*

Definition 10 (Aperiodic)**.** *An irreducible nonnegative matrix is* aperiodic *if* $\gamma(\mathbf{A}) = 1$*[23, Definition 1.6].*

Theorem 11 ([23, Theorem 1.4])**.** *An aperiodic irreducible nonnegative matrix is primitive .*

3. Results

We wish to know the properties of $\mathbf{A}$ that are necessary and sufficient to yield Property I. First it is shown that irreducibility of $\mathbf{A}$ is a necessary condition.

Theorem 12. *Reducible nonnegative matrices never have Property I.*

Proof. This is established by constructing $\mathbf{C}$ and $\mathbf{D}$ such that $\mathbf{C}-\mathbf{D}$ is nonscalar but (3) holds.

The spectrum of a reducible matrix is the union of the spectra of the irreducible diagonal block matrices of its Frobenius normal form [15, p. 29-11]. Its spectral radius is thus the maximum of the spectral radii of these diagonal block matrices.

The Frobenius normal form of a reducible matrix may be represented as a partition of the indices into disjoint nonempty sets $\mathcal{F}_1, \ldots, \mathcal{F}_\nu$ where $\nu \geq 2$. So $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_\nu = \{1, 2, \ldots, n\}$. The irreducible diagonal block matrices are $\mathbf{A}_1, \ldots, \mathbf{A}_\nu$, each of them being principal submatrices of $\mathbf{A}$, where $\mathbf{A}_k := [A_{ij}]_{i,j\in\mathcal{F}_k}$. Thus $r(\mathbf{A}) = \max_{k=1,\ldots,\nu} r(\mathbf{A}_k)$.

In terms of the Frobenius normal form of $e^{\mathbf{D}}\mathbf{A}$,

$$r(e^{\mathbf{D}}\mathbf{A}) = \max_{k=1,\ldots,\nu} r(e^{\mathbf{D}_k}\mathbf{A}_k),$$

where $e^{\mathbf{D}_k}\mathbf{A}_k := [e^{D_i}A_{ij}]_{i,j\in\mathcal{F}_k}$. Let h be one of the maximal blocks, i.e. where $r(e^{\mathbf{D}_h}\mathbf{A}_h) = r(e^{\mathbf{D}}\mathbf{A})$. Now, construct $\mathbf{C}$ from $\mathbf{D}$ thus:

$$\begin{cases} \mathbf{C}_h = \mathbf{D}_h + c_h\mathbf{I}_h & \\ \mathbf{C}_k = \mathbf{D}_k & k \neq h. \end{cases}$$

where $c_h > 0$. Clearly $\mathbf{C}-\mathbf{D}$ is not scalar. For block h, (14) becomes

$$\begin{aligned} \log r(\mathbf{e}^{[(1-t)\mathbf{C}_h + t\mathbf{D}_h]}\mathbf{A}_h) &= \log r(e^{[(1-t)(\mathbf{D}_h + c_h\mathbf{I}_h) + t\mathbf{D}_h]}\mathbf{A}_h) \\ &= \log r(e^{(1-t)c_h}\ e^{\mathbf{D}_h}\mathbf{A}_h) = (1-t)c_h + \log r(e^{\mathbf{D}_h}\mathbf{A}_h) \\ &= (1-t)(c_h + \log r(e^{\mathbf{D}_h}\mathbf{A}_h)) + t\log r(e^{\mathbf{D}_h}\mathbf{A}_h) \\ &= (1-t)\log r(e^{(\mathbf{D}_h + c_h\mathbf{I}_h)}\mathbf{A}_h) + t\log r(e^{\mathbf{D}_h}\mathbf{A}_h) \\ &= (1-t)\log r(e^{\mathbf{C}_h}\mathbf{A}_h) + t\log r(e^{\mathbf{D}_h}\mathbf{A}_h). \end{aligned} \quad (6)$$

Thus equality holds in (14) for block h. Since $c_h > 0$, for all $k \neq h$ and $t \in [0,1]$,

$$r(\mathbf{e}^{[(1-t)\mathbf{C}_h + t\mathbf{D}_h]}\mathbf{A}_h) = (1-t)c_h + \log r(e^{\mathbf{D}_h}\mathbf{A}_h) \geq r(e^{\mathbf{D}_h}\mathbf{A}_h) \geq r(e^{\mathbf{D}_k}\mathbf{A}_k),$$

so block h remains a maximal block for all $t \in [0,1]$, hence

$$r(\mathbf{e}^{[(1-t)\mathbf{C}_h + t\mathbf{D}_h]}\mathbf{A}_h) = r(\mathbf{e}^{[(1-t)\mathbf{C} + t\mathbf{D}]}\mathbf{A}).$$

Thus, the equality (6) implies the equality (3). Since (3) holds even though $\mathbf{C} - \mathbf{D}$ is nonscalar, $\mathbf{A}$ does not have Property I. □

The principal tool to be used next is the set of general necessary and sufficient conditions found by Nussbaum [21, Theorem 1.1, pp. 63–68] for strict log-convexity of the spectral radius of irreducible nonnegative matrices over certain forms of variation. Nussbaum [21, Remark 1.2, pp. 69–70] applies these methods to the particular case of Theorem 4.2 of [10].

So as to be self-contained, relevant excerpts are presented here of Nussbaum's Theorem 1.1 [21], which subsumes the theorems in [18], [6], and [10, Theorems 4.1, 4.2]. The excerpts also include the relevant parts of Nussbaum's proof.

Theorem 13 (Nussbaum [21], Theorem 1.1 Excerpt)**.**

Let $\mathbf{A}$ and $\mathbf{B}$ be nonnegative irreducible $n \times n$ matrices. Let $\mathbf{a}$ and $\mathbf{b}$ be the Perron vectors of $\mathbf{A}$ and $\mathbf{B}$, so $\mathbf{A}\mathbf{a} = r(\mathbf{A})\mathbf{a}$ and $\mathbf{B}\mathbf{b} = r(\mathbf{B})\mathbf{b}$. Let $\mathbf{D}_\mathbf{a}$ and $\mathbf{D}_\mathbf{b}$ refer to the diagonal matrices whose diagonal elements are from the vectors $\mathbf{a}$ and $\mathbf{b}$, respectively.

Define the following 'log-convex combination' of two $n \times n$ matrices:

$$\mathbf{A}^{(1-t)} \circ \mathbf{B}^{(t)} := [A_{ij}^{1-t} B_{ij}^t]_{i,j=1}^n. \tag{7}$$

and of two n-vectors:

$$\mathbf{a}^{(1-t)} \circ \mathbf{b}^{(t)} := [a_i^{1-t}\, b_i^t]_{i=1}^n. \tag{8}$$

Then for all $t \in [0,1]$,

$$r\big(\mathbf{A}^{(1-t)} \circ \mathbf{B}^{(t)}\big) \leq r(\mathbf{A})^{1-t}\ r(\mathbf{B})^t,$$

with equality for some $t \in (0,1)$ if and only if

$$\mathbf{B} = \frac{r(\mathbf{B})}{r(\mathbf{A})}\mathbf{E}^{-1}\mathbf{A}\mathbf{E}, \tag{9}$$

where $\mathbf{E} := \mathbf{D}_\mathbf{a}\mathbf{D}_\mathbf{b}^{-1}$, in which case equality holds for all $t \in [0,1]$.

Nussbaum's proof. The product of the log-convex combinations $\mathbf{A}^{(1-t)} \circ \mathbf{B}^{(t)}$ and $\mathbf{a}^{(1-t)} \circ \mathbf{b}^{(t)}$ manifests Hölder's inequality. For each $i = 1, \ldots, n$:

$$\begin{aligned}\sum_{j=1}^n (A_{ij}a_j)^{1-t}(B_{ij}b_j)^t &\leq \left(\sum_{j=1}^n A_{ij}a_j\right)^{1-t}\left(\sum_{j=1}^n B_{ij}b_j\right)^t \\ &= (r(\mathbf{A})\, a_i)^{1-t}\,(r(\mathbf{B})\, b_i)^t,\end{aligned} \tag{10}$$

or, in vector form,

$$(\mathbf{A}^{(1-t)} \circ \mathbf{B}^{(t)})(\mathbf{a}^{(1-t)} \circ \mathbf{b}^{(t)}) \leq r(\mathbf{A})^{1-t} r(\mathbf{B})^t \ (\mathbf{a}^{(1-t)} \circ \mathbf{b}^{(t)}), \qquad (11)$$

with equality for some $t \in (0, 1)$ if and only if, for each i, the terms in each sum on the right of (10) are proportional, i.e. there exists γ_i such that

$$B_{ij} b_j = \gamma_i A_{ij} a_j, \qquad j = 1, \ldots, n. \qquad (12)$$

Summation over j in (12) gives

$$\sum_{j=1}^{n} B_{ij} b_j = r(\mathbf{B})\, b_i = \gamma_i \sum_{j=1}^{n} A_{ij} a_j = \gamma_i\, r(\mathbf{A})\, a_i,$$

hence γ_i is solved:

$$\gamma_i = \frac{r(\mathbf{B})}{r(\mathbf{A})} \frac{b_i}{a_i}.$$

With this solution, the equality conditions (12) can be rewritten as

$$B_{ij} = \frac{r(\mathbf{B})}{r(\mathbf{A})} \frac{b_i}{a_i} A_{ij} \frac{a_j}{b_j}, \qquad i, j = 1, \ldots, n, \qquad (13)$$

which is the derivation for (9).

The desired term $r(\mathbf{A}^{(1-t)} \circ \mathbf{B}^{(t)})$ emerges from application of the following Subinvariance theorem to (11).

Theorem 14 (Subinvariance [23, Theorem 1.6, p. 23])**.** *For any irreducible nonnegative matrix* $\mathbf{F}$ *and nonnegative vector* $\mathbf{y}$*, if* $\mathbf{F}\mathbf{y} \leq s\mathbf{y}$*, then* $\mathbf{y} > \mathbf{0}$ *and* $r(\mathbf{F}) \leq s$*, with equality if and only if* $\mathbf{F}\mathbf{y} = r(\mathbf{F})\, \mathbf{y}$*.*

In the case here, $\mathbf{F} = \mathbf{A}^{(1-t)} \circ \mathbf{B}^{(t)}$, $\mathbf{y} = \mathbf{a}^{(1-t)} \circ \mathbf{b}^{(t)}$, and $s = r(\mathbf{A})^{1-t}\, r(\mathbf{B})^t$. Therefore

$$r(\mathbf{A}^{(1-t)} \circ \mathbf{B}^{(t)}) \leq r(\mathbf{A})^{1-t}\, r(\mathbf{B})^t,$$

with equality for some $t \in (0, 1)$ if and only if

$$(\mathbf{A}^{(1-t)} \circ \mathbf{B}^{(t)})\ (\mathbf{a}^{(1-t)} \circ \mathbf{b}^{(t)}) = r(\mathbf{A})^{1-t}\, r(\mathbf{B})^t\ (\mathbf{a}^{(1-t)} \circ \mathbf{b}^{(t)})$$

which is precisely equality in (10), whose conditions are given by (13), in which case equality holds for all $t \in [0, 1]$. □

If we let $\mathbf{A}$ and $\mathbf{B}$ in Theorem 13 be substituted by matrices $e^{\mathbf{C}}\mathbf{A}$ and $e^{\mathbf{D}}\mathbf{A}$, $\mathbf{C}$ and $\mathbf{D}$ being diagonal matrices, we obtain:

Corollary 15 (Nussbaum's Remark 1.2 [21]).

Let $\mathbf{A}$ *be an* $n \times n$ *irreducible nonnegative matrix, and* $\mathbf{C}, \mathbf{D} \in \mathcal{D}_n$ *be diagonal matrices. Then for all* $t \in [0, 1]$,

$$\log r(e^{[(1-t)\mathbf{C}+t\mathbf{D}]}\mathbf{A}) \leq (1-t) \log r(e^{\mathbf{C}}\mathbf{A}) + t \log r(e^{\mathbf{D}}\mathbf{A}), \tag{14}$$

with equality for some $t \in (0, 1)$ *if and only if there exists a positive diagonal matrix* $\mathbf{E} \in \mathcal{D}_n$, *and* $\alpha > 0$, *such that*

$$e^{\mathbf{D}}\mathbf{A} = \alpha\, \mathbf{E}^{-1} e^{\mathbf{C}} \mathbf{A}\mathbf{E}, \tag{15}$$

or in terms of matrix elements,

$$e^{D_i} A_{ij} = \alpha E_i^{-1} e^{C_i} A_{ij} E_j, \qquad i, j = 1, \ldots, n. \tag{16}$$

With this machinery in place, we are ready to analyze Property I. Define $L_i := \log E_i$ and $\mathbf{\Delta} := \mathbf{D} - \mathbf{C}$, i.e. $\Delta_i := D_i - C_i$. Then (16) is equivalent to the condition that for each $i, j \in 1, \ldots, n$,

$$A_{ii} = 0, \text{ or } \Delta_i = \log \alpha \qquad j = i; \tag{17}$$

$$A_{ij} = 0, \text{ or } \Delta_i = \log \alpha + L_j - L_i \qquad j \neq i. \tag{18}$$

For $\mathbf{A}$ to have Property I, satisfaction of the set of equalities (17) and (18) must imply that $\mathbf{\Delta} = c\, \mathbf{I}$.

What are necessary and sufficient conditions on $\mathbf{A}$ for (17) and (18) to imply $\mathbf{\Delta} = c\, \mathbf{I}$? We proceed in stages.

Lemma 16. *Property I depends solely upon the sign pattern of* $\mathbf{A}$.

Proof. Whenever $A_{ij} > 0$, A_{ij} cancels out from both sides of (16), so only the sign of A_{ij} (by hypothesis constrained to 0 or +) enters into (17) and (18). □

Lemma 17. *For irreducible* $\mathbf{A}$, *if the equality conditions* (17) *and* (18) *are met, and some* $L_i \neq L_j$, *then* $\mathbf{\Delta} \neq c\, \mathbf{I}$ *for any* $c \in \mathbb{R}$.

Proof. Suppose to the contrary $\mathbf{\Delta} = c\, \mathbf{I}$. This will be shown to imply that $L_i = L_j$ for all i, j.

Irreducibility of $\mathbf{A}$ means by Theorem 7 that for any pair $i, j \in \{1, \ldots, n\}$ either $A_{ij} > 0$, or there is a path of positive elements $(A_{i\,h_1}, A_{h_1 h_2}, \ldots, A_{h_p j})$, or both. When $A_{ij} > 0$ then (18) yields $\Delta_i = \log \alpha + L_j - L_i$, and when $A_{ij} = 0$, repeated application of (18) to the path $(A_{i\,h_1}, A_{h_1 h_2}, \ldots, A_{h_p j})$ gives:

$$\begin{aligned} \Delta_i &= \log \alpha + L_{h_1} - L_i, \\ \Delta_{h_1} &= \log \alpha + L_{h_2} - L_{h_1}, \\ &\ldots \\ \Delta_{h_p} &= \log \alpha + L_j - L_{h_p}. \end{aligned} \tag{19}$$

Summing them and applying the hypothesis $\Delta_i = c$ for all i yields

$$\sum_{k \in \{i, h_1, \ldots, h_p\}} \Delta_k = (p+1)c = (p+1)\log\alpha + L_j - L_i. \tag{20}$$

The case where $A_{ij} > 0$ can be accommodated in (20) by letting $p = 0$.

Irreducibility also implies there must be a reverse path of positive elements $(A_{j\,g_1}, A_{g_1 g_2}, \ldots, A_{g_{p'} i})$ from j to i, yielding

$$\sum_{k \in \{j, g_1, \ldots, g_{p'}\}} \Delta_k = (p'+1)c = (p'+1)\log\alpha + L_j - L_i. \tag{21}$$

Summing (20) and (21) yields

$$(p + p' + 2)c = (p + p' + 2)\log\alpha \iff c = \log\alpha. \tag{22}$$

Substitution of $\Delta_i = c = \log\alpha$ in (19) gives

$$L_i = L_{h_1} = L_{h_2} = \cdots = L_{h_p} = L_j.$$

so $L_i = L_j$. Since this must hold for each choice of $i, j \in \{1, \ldots, n\}$, this means that $L_i = L_j$ for all $i, j = 1, \ldots, n$. By contrapositive inference, if some $L_i \neq L_j$, then $\mathbf{\Delta} \neq c\,\mathbf{I}$ for any $c \in \mathbb{R}$. □

3.1. Main Results

Theorem 18 (Necessary and Sufficient Condition for Property I)**.** *For a non-negative matrix* $\mathbf{A}$ *to have Property I it is necessary and sufficient that* $\mathbf{A}$ *and* $\mathbf{A}^\top\mathbf{A}$ *be irreducible.*

Proof. Since reducible $\mathbf{A}$ do not have Property I by Theorem 12, we assume that $\mathbf{A}$ is irreducible, in which case Nussbaum's [21, Corollary 1.2] applies, and will be used combinatorially. We notice that if two non-diagonal elements in a row of $\mathbf{A}$ are positive, e.g. $A_{ij} > 0, A_{ik} > 0$, then (18) gives

$$\begin{aligned} & \Delta_i = \log\alpha + L_j - L_i = \log\alpha + L_k - L_i \\ \implies \quad & L_k = L_j. \end{aligned}$$

Thus, equality relations between the L_i variables are the result of a single row having multiple positive elements $A_{ij} > 0$. The identity of the row is irrelevant to the L_i values that are equated.

This naturally brings us to the bipartite graph associated with $\mathbf{A}$. Let us define both the directed graph and the simple bipartite graph associated with a matrix.

Definition 19 (Associated Directed Graph)**.** *The* directed graph *(also called* digraph*) associated with an $n \times n$ matrix $\mathbf{A}$ consists of a set of n vertices, and a set of directed edges (also called* arcs*), where an edge goes from vertex j to vertex i when $A_{ij} \neq 0$.*

Definition 20 (Associated Bipartite Graph)**.** *The* simple bipartite graph *associated with an $n \times m$ matrix $\mathbf{A}$ consists of a set $\mathcal{X}$ of n vertices corresponding the row indices of the matrix, a set $\mathcal{Y}$ of m vertices corresponding to the column indices, and a set of undirected edges, where an edge goes between $X_i \in \mathcal{X}$ and $Y_j \in \mathcal{Y}$ when $A_{ij} \neq 0$.*

Let us return to the situation in which a row of $\mathbf{A}$ has two positive elements, A_{ij} and A_{ik}. In the bipartite graph associated with $\mathbf{A}$, this means that there are edges between vertices Y_k and X_i, and between X_i and Y_j. In other words, there is a path between vertices Y_k and Y_j passing through X_i. The existence of a path, and thus equality of L_j and L_k, can be conveniently represented as the condition $\sum_{i=1}^n A_{ij} A_{ik} = [\mathbf{A}^\top \mathbf{A}]_{jk} \neq 0$ since $\mathbf{A}$ is nonnegative.

The transitivity of equality means that if there is a path of any length between Y_j and Y_k (going back and forth between the Y_i's and the X_i's), then the set of equality conditions (18) imply $L_j = L_k$. This occurs if and only if there is some integer $m \geq 1$ such that $[(\mathbf{A}^\top \mathbf{A})^m]_{jk} > 0$. When there is some such m_{jk} for each $j, k \in \{1, \ldots, n\}$, this makes $\mathbf{A}^\top \mathbf{A}$ irreducible by Corollary 6.

Therefore, if $\mathbf{A}^\top \mathbf{A}$ is irreducible in addition to $\mathbf{A}$ being irreducible, the equality conditions (17) and (18) imply $L_i \equiv L$ for all $i = 1, \ldots, n$ (thus $\mathbf{E}$ in (15) is a scalar matrix), hence $\Delta_i = \log \alpha$ for all i, so $\mathbf{\Delta} = \mathbf{D} - \mathbf{C} = \log \alpha\, \mathbf{I}$, satisfying (4), hence $\mathbf{A}$ has Property I. The sufficient part in the theorem is thus proven.

The necessary part means that if $\mathbf{A}^\top \mathbf{A}$ is reducible, then $\mathbf{A}$ does not have Property I. This means that the equality conditions (16) can be met even while $\mathbf{\Delta} \neq c\, \mathbf{I}$ for any $c \in \mathbb{R}$.

To show this, let $\mathbf{A}^\top \mathbf{A}$ be reducible. Then there exist j and k such that $[(\mathbf{A}^\top \mathbf{A})^m]_{jk} = 0$ for all integers $m \geq 1$. For this pair j and k there is no $m \geq 1$ that gives positive $[(\mathbf{A}^\top \mathbf{A})^m]_{jk}$. Hence there is no implication from the set of equality conditions (18) that $L_j = L_k$. Thus we are free to set $L_j \neq L_k$ and still meet (18). From Lemma 17, this implies that $\mathbf{\Delta} \neq c\, \mathbf{I}$ for any $c \in \mathbb{R}$. Thus the equality conditions (17) and (18) do not require $\mathbf{\Delta} = c\, \mathbf{I}$, so $\mathbf{A}$ does not have Property I. □

Application of Theorem 18 allows Theorem 4.2 of [10] to be sharpened as follows.

Theorem 21 (Sharpening of Friedland's Theorem 4.2 [10])**.**

Let $\mathcal{D}_n$ be the set of $n \times n$ real-valued diagonal matrices. Let $\mathbf{A}$ be a fixed $n \times n$ non-negative matrix having a positive spectral radius.

Then:

1. *for each $\mathbf{C}, \mathbf{D} \in \mathcal{D}_n$, $t \in (0, 1)$,*
$$\log r(e^{(1-t)\mathbf{C}+t\mathbf{D}} \mathbf{A}) \leq (1-t) \log r(e^{\mathbf{C}} \mathbf{A}) + t \log r(e^{\mathbf{D}} \mathbf{A}); \qquad (23)$$

2. *if* $\mathbf{D}_1 - \mathbf{D}_2$ *is scalar, equality holds in* (23)*;*
3. *the following are equivalent:*
   (a) $\mathbf{A}$ *and* $\mathbf{A}^\top \mathbf{A}$ *are irreducible;*
   (b) *equality holds in* (23) *only if* $\mathbf{D}_1 - \mathbf{D}_2$ *is scalar;*
   (c) *strict inequality holds in* (23) *for all pairs* $\mathbf{D}_1, \mathbf{D}_2 \in \mathcal{D}_n$ *for which* $\mathbf{D}_1 - \mathbf{D}_2$ *is nonscalar.*

The condition that $\mathbf{A}^\top \mathbf{A}$ be irreducible in order to yield strict inequality also arises in Lemmas 3, 4 and 5 of Cohen et al. [8]. It is notable that they arrive at this condition through analytic means, rather than matrix-combinatorial path used here. Specifically, $\mathbf{A}^\top \mathbf{A}$ enters through the matrix norm $\|\mathbf{A}\| := r(\mathbf{A}^* \mathbf{A})$, where the complex conjugate $\mathbf{A}^* = \mathbf{A}^\top$ when $\mathbf{A}$ is real.

In their Lemmas 3, 4 and 5, the condition that $\mathbf{A}^\top \mathbf{A}$ be irreducible for strict inequality is accompanied by the condition that $\mathbf{A}^2$ also be irreducible. In Theorem 23, to follow, we shall see that irreducibility of both $\mathbf{A}^2$ and $\mathbf{A}^\top \mathbf{A}$ is equivalent to irreducibility of both $\mathbf{A}$ and $\mathbf{A}^\top \mathbf{A}$. It may therefore make sense to call such matrices *two-fold irreducible*.

Definition 22 (Two-fold Irreducibility)**.** *A nonnegative square matrix* $\mathbf{A}$ *is called* two-fold irreducible *if* $\mathbf{A}$ *and* $\mathbf{A}^\top \mathbf{A}$ *are irreducible.*

Two-fold irreducibility therefore defines the underlying condition that is necessary and sufficient for strict inequality in Theorem 21 and in Lemmas 3, 4 and 5 of [8].

Theorem 23 (Two-fold Irreducibility)**.** *For a nonnegative square matrix* $\mathbf{A}$*, the following are equivalent:*

1. $\mathbf{A}$ *and* $\mathbf{A}^\top \mathbf{A}$ *are irreducible;*
2. $\mathbf{A}^2$ *and* $\mathbf{A}^\top \mathbf{A}$ *are irreducible;*
3. $\mathbf{A}$ *and* $\mathbf{A}\mathbf{A}^\top$ *are irreducible;*
4. $\mathbf{A}^2$ *and* $\mathbf{A}\mathbf{A}^\top$ *are irreducible;*
5. $\mathbf{A}$*,* $\mathbf{A}^2$*,* $\mathbf{A}^\top \mathbf{A}$*, and* $\mathbf{A}\mathbf{A}^\top$ *are irreducible;*
6. *the directed graph of* $\mathbf{A}$ *is strongly connected and the simple bipartite graph of* $\mathbf{A}$ *is connected.*

Proof. If $\mathbf{A}^2$ is irreducible, then $\mathbf{A}$ is irreducible, so irreducibility of $\mathbf{A}^2$ and $\mathbf{A}^\top \mathbf{A}$ implies irreducibility of $\mathbf{A}$ and $\mathbf{A}^\top \mathbf{A}$, and irreducibility of $\mathbf{A}^2$ and $\mathbf{A}\mathbf{A}^\top$ irreducibility of $\mathbf{A}$ and $\mathbf{A}\mathbf{A}^\top$. Conversely, suppose that $\mathbf{A}$ is irreducible and $\mathbf{A}^2$ is reducible. We now apply the following theorem from [5] (restated in [15, p. 29-10]), regarding the index of imprimitivity of $\mathbf{A}$.

Theorem 24 (Brualdi and Ryser Theorem 3.4.5 [5])**.** *Let* $\mathbf{A}$ *be an irreducible, nonnegative matrix with index of imprimitivity* $\gamma \geq 2$*. Let* m *be a positive integer. Then* $\mathbf{A}^m$ *is irreducible if and only if* m *and* γ *are relatively prime.*

The hypothesis that $\mathbf{A}$ is irreducible but $\mathbf{A}^2$ is not implies, by Theorem 24, that 2 and the index of imprimitivity γ are not relatively prime, hence 2 divides γ, which means $\gamma \geq 2$ and $\mathbf{A}$ is imprimitive.

Hence $\mathbf{A}$ may be put into the following cyclic normal form by some permutation matrix $\mathbf{P}$ ([2, Sec. 2.2]; [5, Sec. 3.4]; [20, Sec. 3.33.4]; restated in [15, p. 29-10]):

$$\mathbf{A} = \mathbf{P}^\top \begin{bmatrix} \mathbf{0} & \mathbf{B}_2 \\ \mathbf{B}_1 & \mathbf{0} \end{bmatrix} \mathbf{P}.$$

This yields

$$\mathbf{A}^\top \mathbf{A} = \mathbf{P}^\top \begin{bmatrix} \mathbf{0} & {\mathbf{B}_1}^\top \\ {\mathbf{B}_2}^\top & \mathbf{0} \end{bmatrix} \mathbf{P}\mathbf{P}^\top \begin{bmatrix} \mathbf{0} & \mathbf{B}_2 \\ \mathbf{B}_1 & \mathbf{0} \end{bmatrix} \mathbf{P} = \mathbf{P}^\top \begin{bmatrix} {\mathbf{B}_1}^\top \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & {\mathbf{B}_2}^\top \mathbf{B}_2 \end{bmatrix} \mathbf{P},$$

and

$$\mathbf{A}\mathbf{A}^\top = \mathbf{P}^\top \begin{bmatrix} \mathbf{0} & \mathbf{B}_2 \\ \mathbf{B}_1 & \mathbf{0} \end{bmatrix} \mathbf{P}\mathbf{P}^\top \begin{bmatrix} \mathbf{0} & {\mathbf{B}_1}^\top \\ {\mathbf{B}_2}^\top & \mathbf{0} \end{bmatrix} \mathbf{P} = \mathbf{P}^\top \begin{bmatrix} \mathbf{B}_2{\mathbf{B}_2}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_1{\mathbf{B}_1}^\top \end{bmatrix} \mathbf{P}.$$

The presence of the two $\mathbf{0}$ block matrices makes $\mathbf{A}^\top \mathbf{A}$ and $\mathbf{A}\mathbf{A}^\top$ reducible. Thus for irreducible $\mathbf{A}$, the assumption that $\mathbf{A}^\top \mathbf{A}$ or $\mathbf{A}\mathbf{A}^\top$ are irreducible implies by contrapositive inference that $\mathbf{A}^2$ is irreducible. Thus far it is shown $1 \Longleftrightarrow 2$ and $3 \Longleftrightarrow 4$.

In the bipartite graph of $\mathbf{A}$, irreducibility of $\mathbf{A}^\top \mathbf{A}$ means that the $\mathcal{Y}$ vertices are connected. Irreducibility of $\mathbf{A}$ requires that each row have at least one positive element, and thus each $\mathcal{X}$ vertex is connected to the connected $\mathcal{Y}$ vertices, making the entire bipartite graph connected, in particular the $\mathcal{X}$ vertices. Thus $\mathbf{A}\mathbf{A}^\top$ is irreducible. Similarly, irreducibility of $\mathbf{A}$ requires that each column have at least one positive element, so combined with irreducibility of $\mathbf{A}\mathbf{A}^\top$, the same argument yields that $\mathbf{A}^\top \mathbf{A}$ is irreducible. This gives us $1 \Longleftrightarrow 3$, and ties together in equivalence 1, 2, 3, 4, hence 5. In addition, $1 \Longleftrightarrow 3 \Longrightarrow 6$.

Berman and Grone [1, Lemma 2.1] show that the bipartite graph of $\mathbf{A}$ is connected if and only if $\mathbf{A}\mathbf{A}^\top$ and $\mathbf{A}^\top \mathbf{A}$ are irreducible. Boche and Stanczak [3, Theorem 3] show that for irreducible $\mathbf{A}$ the bipartite graph of $\mathbf{A}$ is connected if and only if $\mathbf{A}\mathbf{A}^\top$ is irreducible. This combined with connectedness of the directed graph of $\mathbf{A}$ gives us $6 \Longrightarrow 1, 3$. The equivalence of all the statements is thus shown. □

Remark 25. The list of equivalent pairs of irreducible matrices in Theorem 23 comprises four of the six possible choices from $\{\mathbf{A}, \mathbf{A}^2, \mathbf{A}^\top \mathbf{A}, \mathbf{A}\mathbf{A}^\top\}$. The two missing pairs are $\{\mathbf{A}^\top \mathbf{A}, \mathbf{A}\mathbf{A}^\top\}$ and $\{\mathbf{A}, \mathbf{A}^2\}$. Irreducibility of $\mathbf{A}^\top \mathbf{A}$ and $\mathbf{A}\mathbf{A}^\top$ does not imply irreducibility of $\mathbf{A}$, as can be seen with the simplest example,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \mathbf{A}^\top \mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \mathbf{A}\mathbf{A}^\top = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Irreducibility of $\mathbf{A}$ and $\mathbf{A}^2$ does not imply irreducibility of $\mathbf{A}^\top \mathbf{A}$, as can be seen with this example:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \mathbf{A}^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \mathbf{A}^\top \mathbf{A} = \mathbf{I}.$$

Remark 26. Shmuel Friedland (personal communication) conjectured $1 \Longrightarrow 3$. Joel E. Cohen (personal communication) pointed out that a *scrambling matrix* is two-fold irreducible if it is irreducible, since by definition [12], a scrambling matrix is a row-stochastic matrix that has, for each pair of rows $i \neq j$, some column k such that $A_{ik} A_{jk} > 0$, so this makes $\mathbf{A}\mathbf{A}^\top$ strictly positive off the diagonal, hence irreducible.

Remark 27. Boche and Stanczak [3] venture into the mathematical territory explored here in pursuit of an optimization problem in wireless network reception. Their results overlap, or are related to, [10, Theorem 4.2], Theorem 21, and Theorem 23. Their Theorem 1 is the same as [10, Theorem 4.2] except that conditions for strict convexity are not addressed. Strict convexity conditions are addressed not for the spectral radius itself, but for the shape of regions $\mathcal{F}$ in the space of diagonal matrices $\mathcal{D}_n$ that yield a bounded spectral radius,

$$\mathcal{F}(\mathbf{A}) := \{\mathbf{D} \colon r(e^{\mathbf{D}} \mathbf{A}) \leq 1\} \subset \mathcal{D}_n,$$

which comprise the feasible solutions to their signal-to-interference ratio optimization problem. They show that regions $\mathcal{F}(\mathbf{A})$ are convex, and strictly convex provided $\mathbf{A}$ and $\mathbf{A}\mathbf{A}^\top$ are irreducible. The diagonal matrices $\mathbf{D}$ that they consider [3, Appendix A, p. 1516] are not entirely general, but fall within a set $\mathbf{D} \in \mathcal{S}(\mathbf{A})$ derived as

$$\mathcal{S}(\mathbf{A}) := \left\{ \mathbf{diag}\left[\log \frac{x_i}{[\mathbf{A}\mathbf{x}]_i}\right]_{i=1}^{n} : \mathbf{x} > \mathbf{0} \right\} \subset \mathcal{D}_n.$$

The core of their proof [3, Eq. (10), (11), and Appendix B, p. 1517] utilizes the Cauchy-Schwartz inequality on an expression similar to Nussbaum's (10), with exponent $t = 1/2$. They also prove the equivalence of items 3 and 6 in Theorem 23, but do not include the irreducibility of $\mathbf{A}^2$ and $\mathbf{A}^\top \mathbf{A}$.

Remark 28. The product $\mathbf{A}^\top \mathbf{A}$ plays an important role in Markov chains. To reprise Remark 26, a column-stochastic scrambling matrix has $\mathbf{A}^\top \mathbf{A}$ strictly positive off the diagonal. Kantrowitz et al. [16, Theorem 2.1] show that a column stochastic matrix $\mathbf{A}$ is *contractive* if and only if $\mathbf{A}^\top \mathbf{A} > \mathbf{0}$; by contractive they mean $\|\mathbf{A}\mathbf{u} - \mathbf{A}\mathbf{v}\|_1 < \|\mathbf{u} - \mathbf{v}\|_1$ for all $\mathbf{u} \neq \mathbf{v}$, $\mathbf{u}, \mathbf{v} \in \mathcal{P}_n$, where $\|\mathbf{u}\|_1 := \sum_{i=1}^n |u_i|$, and $\mathcal{P}_n$ is the set of probability vectors. Further, they show that $\mathbf{A}^m \mathbf{x} \to \mathbf{v}$ as $m \to \infty$ for all $\mathbf{x} \in \mathcal{P}_n$ and some $\mathbf{v} \in \mathcal{P}_n$ if and only if $\mathbf{A}$ is 'eventually scrambling' (my phrase by analogy with 'eventually positive'), i.e. there is some positive integer m such that $(\mathbf{A}^m)^\top \mathbf{A}^m > \mathbf{0}$ [16, Theorem 2.3]. It is notable that the product $\mathbf{A}\mathbf{A}^\top$ does not enter into their results.

3.2. Ancillary Results

The paper is concluded with a number of additional results.

Proposition 29. *Two-fold irreduciblity is monotonic in the sign pattern of a nonnegative matrix* $\mathbf{A}$*, i.e. if* $\mathbf{A}$ *has two-fold irreducibility, then changing an element of* $\mathbf{A}$ *from* 0 *to a positive value maintains two-fold irreducibility.*

Proof. This is immediate since the sign pattern of $\mathbf{A}^\top \mathbf{A}$ is monotonic in the sign pattern of $\mathbf{A}$, and irreducibility is monotonic in the sign pattern of a nonnegative matrix.

This can also be seen through direct examination of (18). From Theorem 18, two-fold irreducibility and therefore Property I implies for equality condition (18) that $L_i = L_j$ for all i, j. Suppose that $A_{ij} = 0$ and we change it to be $A_{ij} > 0$. This adds a new constraint to the equality conditions that $\Delta_i = \log \alpha + L_j - L_i$. However this new equation is already satisfied when $\mathbf{A}$ has Property I, which it does by hypothesis, and so the additional equation has no effect. □

Proposition 30. *Full indecomposability is sufficient but not necessary for two-fold irreducibility.*

Proof. Friedland [10, Theorem 4.2] proved that full indecomposability is sufficient for Property I. To prove that it is not necessary I construct a general example of matrices that have Property I, and thus are two-fold irreducible, but which are partly decomposable. A specific example is illustrated for $n = 5$. Without loss of generality 1 is used for the positive elements. Let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}. \tag{24}$$

$\mathbf{A}$ contains the 5-cycle, $5 \to 1 \to 2 \to 3 \to 4 \to 5$ which makes it irreducible. Application of (18) to the bottom row's 1s produces $L_1 = L_2 = L_3 = L_4$. The top row gives $L_2 = L_5$. Thus all L_i are equal, so $\mathbf{\Delta} = \log \alpha\ \mathbf{I}$. It is easily verified that $(\mathbf{A}^\top \mathbf{A})^2 > \mathbf{0}$, hence $\mathbf{A}^\top \mathbf{A}$ is irreducible.

$\mathbf{A}$ is partly decomposable, however, as can be seen from the permutation matrix $\mathbf{Q}$ that rotates the rows up by one, since $\mathbf{QA}$ has $\mathbf{0}$-submatrices of size k by $5 - k$ for each $k = 1, 2, 3, 4$:

$$\mathbf{QA} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

This example can be extended to any n as follows:

$$\begin{cases} A_{1+(j \mod n),j} > 0, & j = 1, \ldots, n \\ A_{ni} > 0, & \forall i \neq n \\ A_{1,2} > 0, & \\ A_{ij} = 0, & \text{otherwise.} \end{cases} \tag{25}$$

The condition $A_{1+(j \bmod n),j} > 0$ for $j = 1, \ldots, n$ produces an n-cycle, which makes $\mathbf{A}$ irreducible. The condition $A_{n,i} > 0 \ \forall i \neq n$ makes all L_i equal for $i = 1, \ldots, n{-}1$. L_n is brought into the equality with the condition $A_{1,2} > 0$, which in combination with $A_{1,n} > 0$ gives $L_n = L_2$. Therefore, $L_i = L$ for $i = 1, \ldots, n$. Substitution in (18) gives $\Delta_i = \log \alpha$ for $i = 1, \ldots, n$. Therefore the equality conditions (16) imply $\mathbf{\Delta} = \log \alpha \, \mathbf{I}$, so $\mathbf{A}$ has Property I.

To verify that $\mathbf{A}$ constructed according to (25) is partly decomposable, we note that applying a permutation $\mathbf{Q}$ that rotates the rows of $\mathbf{A}$ upward by 1 satisfies $[\mathbf{QA}]_{i,n} = 0, i = 1, \ldots, n{-}1$. This is a $n-1$ by 1 submatrix of zeros, making $\mathbf{A}$ partly decomposable. □

Proposition 31. *Primitivity is necessary but not sufficient for two-fold irreducibility.*

Proof.

Primitivity is necessary for two-fold irreducibility. If $\mathbf{A}$ is irreducible but imprimitive, then there exists a permutation matrix $\mathbf{P}$ such that $\mathbf{PAP}^\top$ is in cyclic normal form:

$$\mathbf{A} = \mathbf{P}^\top \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{B}_\gamma \\ \mathbf{B}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & & & \cdots & & \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{B}_{\gamma-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}_{\gamma-1} & \mathbf{0} \end{bmatrix} \mathbf{P}, \tag{26}$$

where each $\mathbf{0}$ block along the diagonal is a square matrix of zeros, of possibly different orders, while the $\mathbf{B}_h$ and $\mathbf{0}$ blocks off the diagonal are rectangular matrices, and γ is the index of imprimitivity of $\mathbf{A}$ [15, p. 29-10]. This yields

$$\mathbf{A}^\top \mathbf{A} = \mathbf{P}^\top \begin{bmatrix} {\mathbf{B}_1}^\top \mathbf{B}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & {\mathbf{B}_2}^\top \mathbf{B}_2 & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & & & \cdots & & \\ \mathbf{0} & \mathbf{0} & \cdots & {\mathbf{B}_{\gamma-2}}^\top \mathbf{B}_{\gamma-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & {\mathbf{B}_{\gamma-1}}^\top \mathbf{B}_{\gamma-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & {\mathbf{B}_\gamma}^\top \mathbf{B}_\gamma \end{bmatrix} \mathbf{P},$$

which shows $\mathbf{A}^\top \mathbf{A}$ to be reducible. Therefore primitivity is necessary for two-fold irreducibility.

Primitivity is not sufficient for two-fold irreducibility. To show this, a general example is provided by the *Wielandt matrix* [16]. An $n \times n$ primitive matrix

A is generated by taking an n-cycle graph and adding a shortcut edge so that it gains a circuit of length $n-1$. Since the greatest common factor of n and $n-1$ is 1, the adjacency matrix for this strongly connected directed graph is aperiodic, hence it is primitive. The matrix has $n+1$ positive elements. It is specified by

$$A_{3,1} = 1, \qquad A_{ij} = \delta_{i,\,1+(j \mod n)} \text{ otherwise.}$$

In the nth row, for $j = 1, \ldots, n$,

$$[\mathbf{A}^\top \mathbf{A}]_{nj} = \sum_{k=1}^{n} A_{kn} A_{kj} = \sum_{k=1}^{n} \delta_{k,\,1+(n \mod n)}\, \delta_{k,\,1+(j \mod n)}$$
$$= \sum_{k=1}^{n} \delta_{k,\,1}\, \delta_{k,\,1+(j \mod n)} = \delta_{1,\,1+(j \mod n)} = \delta_{jn},$$

so row n has a 1 by $n-1$ submatrix of zeros, making $\mathbf{A}^\top \mathbf{A}$ reducible.

To show a concrete example, we add the shortcut $1 \to 3$ to the cycle $1 \to 2 \to 3 \cdots \to 5 \to 1$:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \text{ giving } \mathbf{A}^\top \mathbf{A} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

hence $\mathbf{A}^\top \mathbf{A}$ is reducible. To summarize, primitivity of **A** is necessary but not sufficient to provide two-fold irreducibility. □

Proposition 31 can be further illustrated with a completely worked-out example. Consider a stochastic matrix whose graph comprises two cycles: $1 \to 2 \to 1$ and $1 \to 3 \to 4 \to 1$:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We confirm **A** is primitive by noting that the index of primitivity is the product of the cycle periods 2 and 3 [15, p. 9-7], $2 * 3 = 6$, and $\mathbf{A}^6 > \mathbf{0}$:

$$\mathbf{A}^6 = \begin{bmatrix} 3/8 & 1/2 & 1/4 & 1/2 \\ 1/4 & 1/8 & 1/4 & 1/8 \\ 1/4 & 1/8 & 1/4 & 1/8 \\ 1/8 & 1/4 & 1/4 & 1/4 \end{bmatrix}.$$

The equality condition (18) for $A_{ij} \neq 0$, $j \neq i$, is $D_i - C_i = \log \alpha + L_j - L_i$. The five nonzero entries of **A** $(A_{12}, A_{14}, A_{21}, A_{31}, A_{43})$ thus give five constraints as the equality conditions:

$$\begin{aligned}
A_{12}: &\qquad D_1 - C_1 = \Delta_1 = \log\alpha + L_2 - L_1, \\
A_{14}: &\qquad D_1 - C_1 = \Delta_1 = \log\alpha + L_4 - L_1, \\
A_{21}: &\qquad D_2 - C_2 = \Delta_2 = \log\alpha + L_1 - L_2, \\
A_{31}: &\qquad D_3 - C_3 = \Delta_3 = \log\alpha + L_1 - L_3, \\
A_{43}: &\qquad D_4 - C_4 = \Delta_4 = \log\alpha + L_3 - L_4.
\end{aligned}$$

Since there are 5 constraints on 4 variables Δ_i, 4 variables L_i, and variable c, there are at least $4 + 4 + 1 - 5 = 4$ degrees of freedom in any solution. The above system reduces to:

$$\begin{aligned}
L_4 &= L_2, \\
\Delta_1 &= \log\alpha + L_2 - L_1, \\
\Delta_2 &= \log\alpha + L_1 - L_2, \\
\Delta_3 &= \log\alpha + L_1 - L_3, \\
\Delta_4 &= \log\alpha + L_3 - L_2.
\end{aligned}$$

Thus we are free to specify α, L_1, L_2, L_3 while meeting (17) and (18), which includes values that make $\mathbf{\Delta} = \mathbf{D} - \mathbf{C}$ nonscalar, so the primitive $\mathbf{A}$ does not have Property I and is not two-fold irreducible.

A concrete example of nonscalar $\mathbf{\Delta}$ is provided by $\alpha = e^3$, $L_1 = 1$, $L_2 = -1$, and $L_3 = 2$. Let $C_i = 0$ for $i = 1, 2, 3, 4$, so $D_i = \Delta_i$. Then $\mathbf{D} = \mathbf{\Delta} = \mathbf{diag}[(1, 5, 2, 6)] \neq c\,\mathbf{I}$ for any $c \in \mathbb{R}$. The equality condition is met in (1) if

$$\phi(t) := (1-t) \log r(e^{\mathbf{C}} \mathbf{A}) + t \log r(e^{\mathbf{D}} \mathbf{A}) - \log r(e^{(1-t)\mathbf{C}+t\mathbf{D}} \mathbf{A}) = 0.$$

Since $\mathbf{C} = \mathbf{0}$, this simplifies to:

$$\phi(t) = t \log r(e^{\mathbf{D}} \mathbf{A}) - \log r(e^{t\,\mathbf{D}} \mathbf{A}).$$

It is readily verified that $\log r(e^{\mathbf{D}} \mathbf{A}) = 3$ and $\log r(e^{t\,\mathbf{D}} \mathbf{A}) = 3t$, hence $\phi(t) = 0$. Thus the equality condition is met in (1) while $\mathbf{A}$ is primitive and $\mathbf{D} - \mathbf{C} \neq c\,\mathbf{I}$ for any $c \in \mathbb{R}$.

The next results apply, in particular, to the transition matrices of reversible Markov chains, which have symmetric sign-patterns.

Proposition 32. *A nonnegative matrix with a symmetric sign pattern is two-fold irreducible if and only if it is primitive.*

Proof. By Proposition 31, primitivity is necessary for two-fold irreducibility. It remains to be proven that primitivity is sufficient here. A matrix $\mathbf{B}$ is two-fold irreducible if and only if its sign pattern matrix, $\mathbf{A}$, is two-fold irreducible. A nonnegative matrix $\mathbf{B}$ with symmetric sign pattern $\mathbf{A}$ by definition means $\mathbf{A} = \mathbf{A}^\top$. Assume $\mathbf{A}$ to be symmetric and primitive. Primitivity means there is some integer $m \geq 1$ such that $\mathbf{A}^m > \mathbf{0}$ [11, Theorem 8, p. 80], so clearly $\mathbf{A}^{2m} > \mathbf{0}$, hence $\mathbf{A}^2$ is primitive as well [11, Corollary 1, p. 82]. Thus $\mathbf{A}^\top \mathbf{A} = \mathbf{A}^2$ is irreducible, hence $\mathbf{B}^\top \mathbf{B}$ is irreducible, making $\mathbf{A}$ and $\mathbf{B}$ two-fold irreducible. □

Corollary 33. *The transition matrix of a reversible ergodic Markov chain is two-fold irreducible.*

Proof. The transition matrix is primitive because the chain is ergodic. Reversibility requires $A_{ij}\pi_j = A_{ji}\pi_i$ [17, Proposition 1.3B], where $\boldsymbol{\pi}$ is the stationary distribution of the chain, which being ergodic, yields $\boldsymbol{\pi} > \mathbf{0}$. Hence $A_{ij} > 0$ if and only if $A_{ji} > 0$, so $\mathbf{A}$ has a symmetric sign pattern and Proposition 32 applies. □

Proposition 34. *An irreducible nonnegative matrix with a symmetric sign pattern is either primitive or cyclic of period* 2*.*

Proof. Let $\mathbf{A}$ be the sign pattern matrix for the matrix $\mathbf{B}$. Clearly both $\mathbf{B}$ and $\mathbf{A}$ share the same properties with respect to being irreducible, primitive, or cyclic. Since $\mathbf{A}$ is irreducible, either $\mathbf{A}$ is primitive, or it is imprimitive with index of imprimitivity $\gamma \geq 2$. By [20, Sec. 3.33.4], permutation matrices exist to put $\mathbf{A}$ into a cyclic normal form as in (26), and when γ is greater than 2, into a non-symmetric cyclic normal form (with sub-diagonal blocks as can be seen in (26)), in which case

$$\mathbf{P}\mathbf{A}\mathbf{P}^\top \neq (\mathbf{P}\mathbf{A}\mathbf{P}^\top)^\top = \mathbf{P}\mathbf{A}^\top\mathbf{P}^\top. \tag{27}$$

But then $\mathbf{A} \neq \mathbf{A}^\top$, contrary to hypothesis. Therefore either $\gamma = 2$, or $\mathbf{A}$ is primitive. □

Proposition 35. *The adjacency matrix of a connected simple graph is primitive if and only if the graph is not bipartite.*

Proof. The adjacency matrix, $\mathbf{A}$, of a connected simple graph is an irreducible nonnegative matrix with a symmetric sign pattern, so Proposition 34 applies. The adjacency matrix $\mathbf{A}$ of a bipartite graph can always be permuted into a cyclic normal form of period 2, hence its period is always divisible by 2. But its period cannot be greater than 2 because then $\mathbf{A}$ could be permuted into a non-symmetric cyclic normal form, contrary to its symmetry. Therefore it is cyclic of period 2 if and only if the graph is bipartite. By Proposition 34, if the adjacency matrix is not bipartite, it is thus primitive. □

Corollary 36. *The adjacency matrix of a connected simple graph is two-fold irreducible if and only if the graph is not bipartite.*

Proof. This is a direct consequence of combining Proposition 32 and Proposition 35. □

Remark 37. Joel E. Cohen (personal communication) wondered how much of a gap there was between Friedland's [10]'s condition that $\mathbf{A}$ be irreducible and have positive diagonal elements, and the condition found here that $\mathbf{A}$ and $\mathbf{A}^\top\mathbf{A}$ be irreducible. He found that the gap was small — only one diagonal element, in the case of an n-cycle permutation matrix augmented with positive diagonal elements: one diagonal element may be set to zero while maintaining the irreducibility of $\mathbf{A}^\top \mathbf{A}$, leaving $2n-1$ positive elements; but two diagonal elements set to zero make $\mathbf{A}^\top \mathbf{A}$ reducible.

The number $2n-1$ can be seen to derive from the requirement that the bipartite graph of $\mathbf{A}$ be connected. The graph has $2n$ vertices, and $2n-1$ edges are required to connect them. Any minimally connected graph must be a tree, since an edge that is part of a cycle may be removed without disconnecting the graph.

The terms *indecomposable* [19] [4, p. 329] and *chainable* [13, 14] have been used to refer to a matrix whose associated bipartite graph is connected. The first use of 'chainable' appears to have been by Sinkhorn and Knopp [24] for square matrices, and is defined to be the case where, for any two nonzero elements $A_{i_1 j_1}, A_{i_k j_k}$, there is a sequence $A_{i_1 j_1}, \ldots, A_{i_k j_k}$ of nonzero elements satisfying $i_t = i_{t+1}$ or $j_t = j_{t+1}$. Hartfiel and Maxson [13] modify 'chainable' to apply to $(0,1)$-matrices of order $m \times n$, excluding matrices with a row or column of all zeros. In their Theorem 1.2 they show that the bipartite graph associated with the matrix is connected if and only if the matrix is chainable. In their Lemma 1.1, they show that $\mathbf{A}$ is chainable if and only if no permutation matrices $\mathbf{P}$ and $\mathbf{Q}$ allow $\mathbf{A}$ to be represented as

$$\mathbf{A} = \mathbf{P} \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix} \mathbf{Q}, \tag{28}$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are square diagonal block matrices of order at least one.

It is this property that Hershkowitz et al. [14, Definition 2.12] use to define chainable matrices. In an earlier paper [19], matrices defined by this property are referred to as 'indecomposable', and this usage is maintained in [4, p. 340]. 'Indecomposable' may cause confusion, however, because a matrix such as (24) is then an 'indecomposable partly-decomposable matrix'. 'Chainable' is free of this seeming contradiction. (Also, 'indecomposable' is also used in some quarters instead of 'irreducible'. Other similarly confusing terminology remains, for example, that irreducible matrices are completely reducible [9, p. 127]).

In the Introduction it was asserted that two-fold irreducibility falls in between full indecomposability and primitivity. This is now shown formally.

Theorem 38. *The irreducible nonnegative square matrices form a nested sequence of subsets defined by the following properties, which are monotonic in the sign pattern:*

$$\{\textit{Fully indecomposable}\} \subset \{\textit{Two-fold irreducible}\} \subset \{\textit{Primitive}\} \subset \{\textit{Irreducible}\}.$$

Proof. The definition of full indecomposability is that no permutation matrices $\mathbf{P}$ and $\mathbf{Q}$ allow the matrix $\mathbf{A}$ to be represented as

$$\mathbf{A} = \mathbf{P} \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_2 \end{bmatrix} \mathbf{Q}, \tag{29}$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are square matrices of order at least one. If no permutation matrices $\mathbf{P}$ and $\mathbf{Q}$ can produce (29) then clearly none can produce (28). Thus, fully indecomposable matrices are chainable. They are also irreducible, which requires $\mathbf{Q} = \mathbf{P}^\top$ in (29). Combined with Proposition 31, we obtain the nesting of matrix classes. The monotonicity in the sign pattern for two-fold irreducible matrices is established in Proposition 29, and holds for the other matrix classes because each is defined by conditions that are monotonic in the sign pattern. □

Proposition 39. *If* $\mathbf{A}$ *is two-fold irreducible but not fully indecomposable, then there is no doubly stochastic matrix with the same sign pattern as* $\mathbf{A}$*.*

Proof. This statement is simply a contrapositive of [24, Lemma 1]: A nonnegative matrix $\mathbf{A}$ is fully indecomposable if and only if it is chainable and has doubly stochastic pattern. Hence, if a matrix is not fully indecomposable, one or the other of the two properties must be violated; since two-fold irreducible matrices are chainable, then the sign pattern must not be doubly stochastic if the matrix is to be partly decomposable. □

Remark 40. The example (24) illustrates Proposition 39. Columns 4 and 5 have a single positive element, which in a doubly stochastic matrix would have to equal 1, but there are other positive elements in the rows of these elements, which in a doubly stochastic matrix would make them less than 1.

A memorable way to characterize chainable matrices is introduced by Sinkhorn and Knopp [24, p. 68], which is that a path can be made between any two nonzero elements by moving from one nonzero element to another as a rook does in chess. Irreducible matrices can be characterized in a corresponding way with the following kind of move:

Proposition 41 (Board Moves for Irreducibility)**.** *For a square matrix, starting with one nonzero element, let a sequence of nonzero elements be generated using moves with the following structure:*

1. *move to the reflection of the element's position across the diagonal;*
2. *move horizontally to a nonzero element.*

A matrix is irreducible if and only there is a sequence of such moves from any nonzero element to any other nonzero element, and every row and every column has a nonzero element. Equivalently, move 2 may be replaced with all vertical moves.

Proof. Starting from nonzero element A_{k_1,k_2}, reflection across the diagonal means going from position (k_1, k_2) to (k_2, k_1). The horizontal move then takes one from (k_2, k_1) to a nonzero element A_{k_2,k_3} if such exists. Reflection takes one to (k_3, k_2), and the next horizontal move takes one to a nonzero element A_{k_3,k_4}, etc.. The sequence of nonzero elements generated by moves 1 and 2 therefore has the form $(A_{k_1k_2}, A_{k_2k_3}, \ldots, A_{k_{p-1}k_p})$.

Suppose that such a sequence exists from any nonzero $A_{i_1j_1}$ to any nonzero $A_{i_2j_2}$ and that every row and every column has at least one nonzero element. Then for any pair (i, j) there are nonzero elements A_{ih} and A_{kj} for some $h, k \in$

$\{1, \ldots, n\}$. Since there is a sequence of nonzero elements between A_{ih} and A_{kj}, the condition for irreducibility in Theorem 7 is met.

Conversely, suppose that $\mathbf{A}$ is irreducible. Since there is a sequence of nonzero elements from every i to every j, there must be a nonzero element in each row i and each column j. Suppose that $A_{i_1 j_1}$ and $A_{i_2 j_2}$ are nonzero. By Theorem 7 there is a path of nonzero elements from j_1 to i_2, $(A_{j_1 k_1}, A_{k_1 k_2}, \ldots, A_{k_p i_2})$. This path joins $A_{i_1 j_1}$ and $A_{i_2 j_2}$ to create $(A_{i_1 j_1}, A_{j_1 k_1}, A_{k_1 k_2}, \ldots, A_{k_p i_2}, A_{i_2 j_2})$, which shows that any pair of nonzero elements $A_{i_1 i_2}$ and $A_{j_1 j_2}$ can be connected by a sequence of nonzero elements as generated by moves 1 and 2.

A sequence of moves on $\mathbf{A}$ is equivalent to a sequence of moves on $\mathbf{A}^\top$ where horizontal moves are replaced by vertical moves for step 2. Since $\mathbf{A}$ is irreducible if and only if $\mathbf{A}^\top$ is irreducible, horizontal moves in step 2 may be replaced by vertical moves. □

Useful reviews of the properties of chainable matrices can be found in [22, Chapter 5] and [4, Chapter 8]. Two significant properties of chainable matrices, pointed out by [13] (see also [22, p. 190]), are that the chainable $n \times n$ matrices comprise a semigroup under matrix multiplication, and furthermore, that this semigroup is a two-sided ideal within the set of $n \times n$ nonnegative matrices with no row or column of zeros. (Two-fold irreducible matrices, however, are not a semigroup, since their products may be reducible, as can easily be verified numerically.) Chainable matrices are also to be found under the rubric of 'transportation polytopes', and in particular are a means to characterize the *nondegenerate polytopes* [4, Chapter 8]. The extreme points of the polytope of square matrices have $2n - 1$ positive elements ([22, p. 160]; [4, p. 340]).

As shown in Theorem 23, the two-fold irreducible matrices that emerge here are the intersection between the chainable and the irreducible nonnegative matrices. Further characterization of the properties of this intersection may prove of value.

Acknowledgements

Thanks go to Joel E. Cohen for sharing with me the open questions from which this paper grew, and for his comments; to Roger Nussbaum for his definitive treatment which provided the key tools used here; to Shmuel Friedland for referring Joel Cohen to me; to each for populating the field with so many fruitful results; and to Laura Marie Herrmann for assistance with the literature search.

References

References

[1] A. Berman, R. Grone, Bipartite completely positive matrices, Proceedings of the Cambridge Philosophical Society 103 (1988) 269–276.

[2] A. Berman, R. J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, 2nd ed., Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1994.

[3] H. Boche, S. Stanczak, Strict convexity of the feasible log-SIR region, IEEE Transactions on Communications 56 (9) (2008) 1511–1518.

[4] R. Brualdi, Combinatorial Matrix Classes, vol. 108 of Encyclopedia of Mathematics and Its Applications, Cambridge University Press, Cambridge, 2006.

[5] R. Brualdi, H. Ryser, Combinatorial Matrix Theory, vol. 39 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1991.

[6] J. E. Cohen, Convexity of the dominant eigenvalue of an essentially nonnegative matrix, Proceedings of the American Mathematical Society 81 (4) (1981) 657–658.

[7] J. E. Cohen, Cauchy inequalities for the spectral radius of products of diagonal and nonnegative matrices, Proceedings of the American Mathematical Society (2012) In press.

[8] J. E. Cohen, S. Friedland, T. Kato, F. P. Kelly, Eigenvalue inequalities for products of matrix exponentials, Linear Algebra and Its Applications 45 (1982) 55–95.

[9] B. Eaves, A. Hoffman, U. Rothblum, H. Schneider, Line-sum-symmetric scalings of square nonnegative matrices, Mathematical Programming Study 25 (Essays in Honor of George B. Dantzig Part II) (1985) 124–141.

[10] S. Friedland, Convex spectral functions, Linear and Multilinear Algebra 9 (1981) 299–316.

[11] F. R. Gantmacher, The Theory of Matrices, vol. 2, Chelsea Publishing Company, New York, 1959.

[12] J. Hajnal, Weak ergodicity in non-homogeneous Markov chains, Proceedings of the Cambridge Philosophical Society 54 (1958) 233–246.

[13] D. J. Hartfiel, C. J. Maxson, The chainable matrix, a special combinatorial matrix, Discrete Mathematics 12 (3) (1975) 245–256.

[14] D. Hershkowitz, U. Rothblum, H. Schneider, Classifications of nonnegative matrices using diagonal equivalence, SIAM Journal of Matrix Analysis and Applications 9 (4) (1988) 455–460.

[15] L. Hogben (ed.), Handbook of Linear Algebra, Chapman and Hall, 2007.

[16] R. Kantrowitz, M. Neumann, T. Ransford, Regularity, scrambling, and the steady state for stochastic matrices, in: K. Jarosz (ed.), Function Spaces in Modern Analysis: Sixth Conference on Function Spaces, May 18-22, 2010, Southern Illinois University, Edwardsville, vol. 547, American Mathematical Society, Providence, RI, 2011, pp. 153–164.

[17] J. Keilson, Markov Chain Models: Rarity and Exponentiality, Springer-Verlag, New York, 1979.

[18] J. F. C. Kingman, A convexity property of positive matrices, The Quarterly Journal of Mathematics 12 (1) (1961) 283–284.

[19] M. V. Menon, H. Schneider, The spectrum of a nonlinear operator associated with a matrix, Linear Algebra Appl 2 (1969) 321–334.

[20] H. Minc, Nonnegative Matrices, John Wiley and Sons, New York, 1988.

[21] R. D. Nussbaum, Convexity and log convexity for the spectral radius, Linear Algebra and Its Applications 73 (1986) 59–122.

[22] V. Sachkov, V. Tarakanov, Combinatorics of nonnegative matrices, vol. 213, Amer Mathematical Society, 2002.

[23] E. Seneta, Non-negative Matrices and Markov Chains, revised ed., Springer-Verlag, New York, 2006.

[24] R. Sinkhorn, P. Knopp, Problems involving diagonal products in nonnegative matrices, Transactions of the American Mathematical Society 136 (1969) 67–75.